JOURNAL OF ALGEBRA 128, 306–320 (1990)

Analytic Spread Modulo an Element and Symbolic Rees Algebras

SAM HUCKABA

Department of Mathematics, Florida State University, Tallahassee, Florida 32306-3027

Communicated by Melvin Hochster

Received January 27, 1988

INTRODUCTION

Let (T, N) be a d-dimensional regular local ring, P a prime ideal of T having height equal $d-1$, and $f \in P$. Within this setting we are interested in computing the analytic spread of P/fT. We have two reasons for seeking this analytic spread. The first is to identify prime ideals in local hypersurface rings whose analytic spreads and heights are the same. The second is to aid in a better understanding of the symbolic Rees algebra, $\bigoplus P^{(n)}t^n$, of P (the ideal $P^{(n)}$ is the nth symbolic power of P).

Our interest in hypersurface rings essentially originated in [Hu2]. There we were interested in determining when powers and symbolic powers are equal for a prime ideal in a local hypersurface ring. For a prime ideal q adjacent to the maximal ideal (i.e., height $(M/q)=1$), the only time such an equality can occur for all large powers is when the analytic spread of q equals the height of q. Equality does not, however, necessarily occur under this analytic spread condition (cf. [Hu2, Example 2.10]).

In order to force equality for all powers, one must make additional assumptions on the associated graded ring of $P/fT=q$ (cf. [Hu2, Theorem 2.1]). Section 2 of the present article contains some general methods for recognizing the analytic spread of P/fT, and Section 3 contains some illustrative examples where the analytic spread of P/fT equals the height of P/fT.

The Noetherian property of the symbolic Rees algebra has created recent interest due to a question of Cowsik. In connection with his result in [C] that $\bigoplus P^{(n)}t^n$ Noetherian implies P is a set-theoretic complete intersection when T is a regular local ring and $\dim(T/P)=1$, Cowsik asked whether $\bigoplus P^{(n)}t^n$ is Noetherian for every prime ideal in a regular local ring. An example due to P. Roberts [Ro] showed the answer is no in general, but the question has continued to draw attention [H1, H2, KR, O, E]. The

0021-8693/90 $3.00

property that the analytic spread of P/fT equals the height of P/fT for some $f \in P$ has a connection with the Noetherian property of $\bigoplus P^{(n)}t^n$. By making certain technical assumptions on f (and the way f sits inside P), and assuming the above analytic spread condition, we are able to prove that $\bigoplus P^{(n)}t^n$ is Noetherian (see Theorem 2.5 for the precise statement). The examples produced in Section 3 are then also applications of Theorem 2.5.

The paper is divided into three sections with Section 1 providing background information, Section 2 the results, and Section 3 the examples. We remark that the techniques developed by C. Huneke in [H1, H2] were most useful for our investigation.

1. Background

Let (R, M) be a commutative Noetherian local ring and let q be a prime ideal of R. The Rees algebra of R, denoted by $R[qt]$, is the graded ring $\bigoplus q^n t^n$. The symbolic Rees algebra of q is the graded ring $\bigoplus q^{(n)}t^n$, where $q^{(n)} = \{r \in R/rs \in q^n \text{ for some } s \in R \backslash q\}$. Notice that $R[qt]$ is a subring of $\bigoplus q^{(n)}t^n$, which in turn is a subring of $R[t]$. By viewing $\bigoplus q^{(n)}t^n$ as a module over $R[qt]$, researchers have investigated the property that $\bigoplus q^{(n)}t^n$ is module finite over $R[qt]$ [H3, S, R]. This finiteness property forces that the symbolic powers of q are "well behaved" in the sense that there exists c for which $q^{(n+c)} \subset q^n$ for all $n \geqslant 1$ (linear equivalence of the q-adic and q-symbolic topologies). A fundamental tool for examining properties of the powers of an ideal is the analytic spread of an ideal, a concept developed by Northcott and Rees [NR]. Recall that if (R, M) is local with R/M infinite and I is an ideal of R, then the analytic spread of I is defined as the Krull dimension of the graded ring $\bigoplus I^n/MI^n$. We will denote the analytic spread of I by $l(I)$. In [S, Corollary 1], Schenzel showed (for R unmixed) that the finite module condition on $\bigoplus q^{(n)}t^n$ over $R[qt]$ is equivalent to the condition that $l(qR_Q) < \text{height}(Q)$ for every prime ideal Q properly containing q (in fact Schenzel does not require that R be local). In [Hu1, Examples 1.5 and 1.6] and again in [Hu2, Example 2.10], the author gave explicit non-trivial examples of prime ideals q satisfying Schenzel's criterion (by non-trivial we mean that $q^n \neq q^{(n)}$ for all large n).

Suppose R is a local hyprsurface ring, that is, $R = T/fT$ where T is a regular local ring and $f \in T$. If q is a prime ideal of R such that $\dim(R/q) = 1$, then Schenzel's analytic spread condition from above reduces to $l(q) = \text{height}(q)$. Furthermore, q has the form $q = P/fT$ for some prime ideal P of T. Knowing that $\bigoplus q^{(n)}t^n$ is module finite over $R[qt]$ need not force that $\bigoplus P^{(n)}t^n$ is module finite over $T[Pt]$ (see Examples 3.1, 3.2, 3.3).

It does, however, imply that P is a set-theoretic complete intersection (that is, P can be generated up to radical by height (P) elements). This is because P/fT has a *reduction* generated by height$(P)-1$ elements, hence is the radical of a height$(P)-1$ generated ideal, and this implies P is the radical of a height(P) generated ideal. Recall that a reduction of an ideal I is an ideal $J \subseteq I$ satisfying an equation $JI^m = I^{m+1}$ for some positive integer m; thus I has the same radical as any of its reductions. Northcott and Rees proved that an ideal I in a local ring having an infinite residue field has a reduction generated by $l(I)$ elements [NR].

While $\bigoplus P^{(n)}t^n$ need not be module finite over $T[Pt]$ when $l(P/fT) =$ height(P/fT), we show in Theorem 2.5 that if $f \in P^{(m)}$ and the *reduction number* of P/fT is at most $m-1$, then $\bigoplus P^{(n)}t^n$ is Noetherian (the reduction number of an ideal I is defined by $r(I) = \min\{n \in Z^+ \cup \{0\} \mid$ there exists a minimal reduction $J \subseteq I$ with $JI^n = I^{n+1}\}$). We obtain a similar result in Theorem 2.4. These results show that certain finiteness conditions on $\bigoplus q^{(n)}t^n$ can sometimes be lifted back to $\bigoplus P^{(n)}t^n$ (in this case the Noetherian property). The examples presented in Section 3 will illustrate both Theorem 2.5 and the conditions required for Schenzel's result.

We close this section by settling on some notation. If I is an ideal of a Noetherian ring R, we denote the associated graded ring of R with respect to $I(\bigoplus I^n/I^{n+1})$ by $G(I, R)$. If $f \in R$ we denote the leading form of f in $G(I, R)$ by f'. Recall that f is said to be integral over I if it satisfies an equation of the form $f^m + a_1 f^{m-1} + \cdots + a_{m-1} f + a_m = 0$ where $a_i \in I^i$. We denote the integral closure of I (i.e., the ideal of elements all of which are integral over I) by I'. All rings are assumed to be commutative and Noetherian. A reference for unexplained notation and terminology is the book by Matsumura [Mat].

2. Results

Let (T, N) be a Noetherian local ring with T/N infinite, P a prime ideal of T, and $f \in P$. Set $R = T/fT$, $M = N/fT$, and $q = P/fT$. Since $\bigoplus (P^n/NP^n)$ is isomorphic to $T[Pt]/NT[Pt]$ it follows that $l(P) =$ $\dim(T[Pt]/NT[Pt])$. In order to calculate $l(q)$, the corresponding isomorphism between $\bigoplus q^{(n)}t^n$ and $R[qt]/MR[qt]$ shows that a computation of $\dim(R[qt]/MR[qt])$ will suffice. By using the T-module isomorphism

$$(P^n, fT)/(NP^n, fT) \cong P^n/(NP^n, fT \cap P^n) = P^n/(NP^n, fT \cap P^n)$$

we can think of $R[qt]/MR[qt]$ as a homomorphic image of $T[Pt]$. More specifically, if $J(f) = \bigoplus (fT \cap P^n)\, t^n$, then $R[qt]/MR[qt] \cong T[Pt]/(N, J(f))\, T[Pt]$. We refer to the ideal $J(f)$ as the ideal of leading forms of f in $T[Pt]$ (that is, the ideal in $T[Pt]$ generated by all elements in $T[Pt]$ of the form rft^n where $r \in T$).

Below we will make some observations about $l(P/fT)$ in the special case that P is generated by analytically independent elements. We first recall the definition of analytically independent.

DEFINITION 2.1. Let (T, N) be a local ring and $x_1, ..., x_d \in T$. If all homogeneous polynomials $F(Z_1, ..., Z_d) \in T[Z_1, ..., Z_d]$, where $Z_1, ..., Z_d$ are indeterminates, satisfy the property that $F(x_1, ..., x_d) = 0$ implies $F \in NT[Z_1, ..., Z_d]$, then $x_1, ..., x_d$ are said to be analytically independent.

Remark. Elements $x_1, ..., x_d \in T$ are analytically independent if and only if $T[It]/NT[It]$ is isomorphic to a polynomial ring in d variables over T/N, where $I = (x_1, ..., x_d)T$. From [NR, Sect. 3, Theorem 3] it follows that if $x_1, ..., x_d$ are analytically independent, then $l(I) = d$ where $I = (x_1, ..., x_d)T$.

Recall that an element g in a Noetherian ring R is said to be superficial of order s for an ideal I of R if $g \in I^s \backslash I^{s+1}$ and there exists a positive integer c for which $(I^n : g) \cap I^c = I^{n-s}$ for all sufficiently large n [ZS, p. 285]. In particular if g' is a regular element in $G(I, R)$, then g is superficial of order equal to its order with respect to I. We will say that an element is superficial for I if it is superficial of order equal to its order with respect to I.

PROPOSITION 2.2. *Let (T, N) be a local domain, P a prime ideal of T, and $f \in P$. Assume T/N is infinite, and f is superficial for P.*

(i) *Then $l(P/fT) \geq l(P) - 1$.*

(ii) *If P is generated by analytically independent elements, then $l(P/fT) = l(P)$ if and only if $J(f) \subseteq NT[Pt]$.*

Proof. (i) Set $R = T/fT$, $M = N/fT$, and $q = P/fT$, and assume $f \in P^m \backslash P^{m+1}$. We must show that $\dim(R[qt]/MR[qt] \geq \dim(T[Pt]/NT[Pt]) - 1$. By the remarks at the beginning of this section it follows that

$$R[qt]/MR[qt] \cong \bigoplus \; (P^n/(NP^n, fT \cap P^n)).$$

In general it holds that

$$fT \cap P^n = f(P^n : f);$$

thus $R[qt]/MR[qt] \cong \oplus (P^n/(NP^n, f(P^n : f)))$. Since f is superficial of order m, there exists a positive integer c such that $(P^n : f) \cap P^c = P^{n-m}$ for n sufficiently large. By the Artin–Rees Theorem [Mat, Proposition (11.E)], for n sufficiently large $(P^n : f) \subset P^c$. Thus, for large n we have

$(P^n : f) = P^{n-m}$. It follows that the Hilbert polynomial for the graded ring $\bigoplus (P^n/(NP^n, f(P^n : f)))$ has the *same degree* as the Hilbert polynomial for the graded ring $\bigoplus (P^n/(NP^n, fP^{n-m}))$. The theory of Hilbert polynomials thus implies that these two graded rings have the same Krull dimension (this follows for example by representing each as the associate graded ring with respect to the maximal ideal of its localization at the maximal homogeneous ideal, then using [Mat, Theorem 17]). If we adopt the convention that $P^n = T$ if $n \leqslant 0$, then $(N, ft^m)\, T[Pt] = \bigoplus (NP^n, fP^{n-m})t^n$; hence $T[Pt]/(N, ft^m)T[Pt] \cong \bigoplus (P^n/(NP^n, fP^{n-m}))$. Stringing together all the equalities and isomorphisms yields

$$\begin{aligned} l(q) &= \dim(R[qt]/MR[qt]) \\ &= \dim(\bigoplus (P^n/(NP^n, f(P^n : f)))) \\ &= \dim(\bigoplus (P^n/(NP^n, fP^{n-m}))) \\ &= \dim(T[Pt]/(N, ft^m)\, T[Pt]). \end{aligned}$$

By applying Krull's principal ideal theorem it follows that $\dim(T[Pt]/(N, ft^m)\, T[Pt]) \geqslant \dim(T[Pt]/NT[Pt]) - 1$. This translates to $l(q) \geqslant l(P) - 1$.

(ii) Since P is generated by analytically independent elements, $NT[Pt]$ is a prime ideal of $T[Pt]$. As we saw earlier,

$$l(P/fT) = \dim(T[Pt]/(N, J(f))\, T[Pt]).$$

It now follows immediately that $l(P/fT) = l(P)$ if and only if $J(f) \subseteq NT[Pt]$. ∎

COROLLARY 2.3. *Let (T, N) be a local domain with T/N infinite, let P be a prime ideal of T, and let $f \in P$. Assume f' is a regular element in $G(P, T)$ and $f \in P^m \backslash P^{m+1}$. Then $l(P/fT) = l(P) - 1$ if and only if f is part of a minimal generating set for P^m. Otherwise $l(P/fT) = l(P)$.*

Proof. Observe that $(N, J(f))\, T[Pt] = (N, ft^m)\, T[Pt]$; hence $J(f) \subseteq NT[Pt]$ if and only if $ft^m \in NT[Pt]$. Furthermore, f is part of a minimal generating set for P^m if and only if $ft^m \notin NT[Pt]$. The result now follows from Proposition 2.2. ∎

Proposition 2.2 gives the whole story for $l(P/fT)$ when P is generated by analytically independent elements and f is superficial. As an example for Corollary 2.3, we could take T to be a three-dimensional regular local ring and P a height two complete intersection prime ideal of T. Then any element $f \in P$ satisfies the property that f' is regular in $G(P, T)$, because $G(P, T)$ is a domain. Therefore $l(P/fT) = 1$ if and only if f is part of a minimal generating set for P^m where $f \in P^m \backslash P^{m+1}$ (we are using here that $l(P) = 2$ if P is a complete intersection). The setting of this example was considered in [Hu2], where other properties of P/fT were also studied. On the other hand if P is not a complete intersection then $l(P/fT) = l(P) - 1$ if f is superficial for P (that is, the condition in Proposition 2.2(ii) never occurs). This follows because $l(P) = \dim(T)$ in this case [CN], and the analytic spread of an ideal is always bounded by the dimension of the ring; hence $l(P/fT) \leqslant \dim(T/fT) = \dim(T) - 1 = l(P) - 1$. Applying Proposition 2.2(i) then gives $l(P) = l(P) - 1$.

If $f \in P$ is not superficial then the description of $l(P/fT)$ is more complicated. The trouble is that we no longer have just two choices for $l(P/fT)$; hence the computation requires a more thorough understanding of $J(f)$. This in itself is difficult. In Section 3 we will present examples computing $l(P/fT)$ when f is not superficial (the examples will also show that it is possible to have $l(P/fT) < l(P) - 1$).

We now turn to an analysis of the Noetherian Property of the symbolic Rees algebra of P. Below we will prove a result revealing a connection between knowledge of $l(P/fT)$ and the finite generation of $\bigoplus P^{(n)}t^n$ over T. Before doing so we prove a related result which gives somewhat of a general criterion for $\bigoplus P^{(n)}t^n$ to be Noetherian. Recall that a local ring R is said to be analytically unramified if its completion $\hat{R}$ is reduced, and it is said to be quasi-unmixed if it satisfies the property that $\dim(\hat{R}/q) = \dim(\hat{R})$ for every minimal prime q of $\hat{R}$. For the proof of our first result we need the following observation.

LEMMA. *Let (T, N) be a local ring having an infinite residue field and let I be an ideal of T. If $J = (g_1, g_2, ..., g_d)T$ is a reduction of I, then $J_m = (g_1^m, g_2^m, ..., g_d^m)T$ is a reduction of I^m for any positive integer m.*

Proof. By assumption there exists n such that $JI^n = I^{n+1}$. For any positive integer m we have $J^m I^{nm} = (JI^n)^m = (I^{n+1})^m = (I^m)^{n+1}$; hence J^m is a reduction of I^m.

Thus, it suffices to show that J_m is a reduction of J^m, and this will follow if we show that J^m is integral over J_m (see [NR, Sect. 7, Theorem 3] if I contains a regular element, and use [L, Lemma 1.1] in the general case). To show that J^m is integral over J_m we need to prove that the mixed terms $g_i^c g_j^{m-c}$ (where $i \neq j$ and $c > 0$) are integral over J_m. But this is clear since $(g_i^c g_j^{m-c})^m = (g_i^m)^c (g_j^m)^{m-c} \in J_m^m$. ∎

THEOREM 2.4. *Let (T, N) be a d-dimensional quasi-unmixed analytically unramified local domain having an infinite residue field, and let P be a prime ideal of T such that* height$(P) = d - 1$. *If there exist elements $f_1, ..., f_{d-1} \in P$, a positive integer m, and a reduction $(g_1, ..., g_d)T$ of P, satisfying the* *conditions that* $(g_1, ..., g_d)P^{m-1} = P^m, f_1, ..., f_{d-1} \in P^{(m)}$, *and* $g_i^m \in (f_1, ..., f_{d-1})T$, *then* $\bigoplus P^{(n)}t^n$ *is Noetherian.*

Proof. Set $J = (f_1, ..., f_{d-1})T$. Our plan is to show that $l(P^{(m)}) = d-1$, and then apply a theorem due to Katz and Ratliff. Since $J \subset P^{(m)}$, we have that $P^m T_P = (P^m, J)T_P$. We claim that JT_P is a reduction of $P^m T_P$. Set $I = (g_1, ..., g_d)T$. Then $(P^m, J)T_P = (IP^{m-1}, J)T_P$ by the reduction number assumption on I and P. Now we compute $(P^m)^m T_P$. Using the above equality,

$$\begin{aligned}(P^m)^m T_P &= (IP^{m-1}, J)^m T_P\\ &= (I^m(P^{m-1})^m, I^{m-1}(P^{m-1})^{m-1}J, ..., IP^{m-1}J^{m-1}, J^m)T_P\\ &= (I^m(P^{m-1})^m, J[IP^{m-1}, J]^{m-1})T_P\\ &= (I^m(P^m)^{m-1}, J(P^m)^{m-1})T_P = (I^m, J)(P^m)^{m-1}T_P.\end{aligned}$$

Therefore $(I^m, J)T_P$ is a reduction of $P^m T_P$. By the lemma, $I_m = (g_1^m, ..., g_d^m)T$ is a reduction of I^m; thus $(I_m, J)T_P$ is a reduction of $(I^m, J)T_P$ [NR, Sect. 7, Lemma 1]. It follows that $(I_m, J)T_P$ is a reduction of $P^m T_P$, and this implies JT_p is a reduction of $P^m T_P$ because $I_m \subseteq J$. We next claim that J is a reduction of $P^{(m)}$, a fact which will force $l(P^{(m)}) = d-1$. Set $A = J'$. Since the integral closure operation commutes with localization, we have the equalities $J'T_P = (JT_P)' = (P^m T_P)'$ (since JT_P is a reduction of $P^m T_P) = (P^m)'T_P = AT_P$. Thus we have the chain of ideals $JT_P \subseteq P^m T_P \subseteq AT_P$. Since height$(J) = d-1$ (because the radical of J is P) and J is generated by $d-1$ elements, it follows from [NR, Sect. 4, Theorem 5] that $l(J) = d-1$. Thus $l(A) = d-1 < \text{height}(N)$. By applying [M, Proposition 4.1] it follows that N is not an associated prime of $(A^n)'$ for all large n, and therefore since $A' = A$ we have $N \notin \text{Ass}(T/A)$ by [M, Proposition 3.9]. But A has radical P, so it follows that $\text{Ass}(T/A) = \{P\}$. By contracting the chain $JT_P \subseteq P^m T_P \subseteq AT_P$ back to T we thus obtain $J \subseteq P^{(m)} \subseteq A$. Since $J' = A$ if follows from [NR, Sect. 7, Theorem 3] that J is a reduction of $P^{(m)}$. This shows that $l(P^{(m)}) = d-1$. By applying [KR, Theorem A] we obtain that $\bigoplus P^{(n)}t^n$ is Noetherian. ∎

THEOREM 2.5. *Let* (T, N) *be a d-dimensional quasi-unmixed analytically unramified local domain having an infinite residue field, and P a prime ideal of T with height* $(P) = d-1$. *If there exists* $f \in P^{(m)}$, *m a positive integer, such that* $l(P/fT) = d-2$ *and* $r(P/fT) \leqslant m-1$, *then* $\bigoplus P^{(n)}t^n$ *is Noetherian.*

Proof. The proof is similar to the proof of Theorem 2.4. We will show that $l(P^{(m)}) = d-1$, and then apply [KR, Theorem A]. Let $J = (f_1, ..., f_{d-2})T$ be an ideal of T such that $J_1 = (\bar{f}_1, \bar{f}_2, ..., \bar{f}_{d-2})$ is a minimal reduction of P/fT where $\bar{f}_i$ is the image of f_i modulo fT, and

$r_{J_1}(P/fT) \leqslant m-1$ i.e., $J_1 q^{m-1} = q^m$ where $q = P/fT$). Then $(JP^{m-1}, f) = (P^m, f)$. Since $f \in P^{(m)}$, $(JP^{m-1}, f)T_P = P^m T_P$. By a computation similar to that in Theorem 2.4 it holds that $(P^m)^m T_P = (J^m, f)(P^m)^{m-1} T_P$; hence $(J^m, f)T_P$ is a reduction of $P^m T_P$. Since $l(J^m) = l(J) = d-2$, there exists $b_1, ..., b_{d-2} \in J^m$ such that $I = (b_1, ..., b_{d-2})T$ is a minimal reduction of J^m. By [NR, Sect. 7, Lemma 1], $(I, f)T_P$ is a reduction of $P^m T_P$. We claim that $(I, f)T$ is a reduction of $P^{(m)}$. As in the proof of Theorem 2.4 we set $A = (I, f)'$. Since the radical of (I, f) is P, the radical of A is also P. By using [M, Proposition 4.1] and [M, Proposition 3.9] again (as in the proof of Theorem 2.4) it follows that $\mathrm{Ass}(T/A) = \{P\}$ so that $AT_P \cap T = A$. Consider the chain of ideals $(I, f)T_P \subseteq P^m T_P \subseteq (P^m)' T_P = (I, f)' T_P = AT_P$. By contracting this chain we obtain $(I, f) \subseteq P^{(m)} \subseteq A$; thus (I, f) is a reduction of $P^{(m)}$. Therefore $l(P^{(m)}) = d-1$ so that by [KR, Theorem A], $\bigoplus P^{(n)}t^n$ is Noetherian. ∎

We make several remarks concerning Theorems 2.4 and 2.5

Remarks 2.6. (1) usefulness of Theorem 2.4 is not yet entirely clear to us. We do note, however, that it can be applied to the following concrete example. Let $T = K[[U, V, W]]/(U^a - V^b h) K[[U, V, W]] = K[[x, y, z]]$ where $a \geqslant b$ are positive integers, U, V, W are indeterminates, and h is an element of $K[[U, V, W]]$ that is not in $(U, V) K[[U, V, W]]$. Assume $U^a - V^b h$ is irreducible so that T is a domain. Set $P = (x, y)T$. Then $y^b \in P^{(a)}$, $x^a, y^a \in y^b T$, and $(x, y)P^{a-1} = P^a$. Therefore by Theorem 2.4, $\bigoplus P^{(n)}t^n$ is Noetherian. It appears that Theorem 2.4 might be useful for considering prime ideals generated by analytically independent elements (as in this example).

(2) We mention that regular local rings and complete local domains are examples of local domains that satisfy the assumptions on T in Theorems 2.4 and 2.5.

(3) For Theorem 2.5 we do not know whether the assumption $r(P/fT) \leqslant m-1$ can be removed. More specifically, if we merely assume $l(P/fT) = d-2$ does it follow that $\bigoplus P^{(n)}t^n$ is Noetherian?

(4) In the three-dimensional Cohen–Macaulay case, Theorem 2.5 is apparently closely related to Theorem 1.4 of [H1] (we will see this more emphatically when we examine some examples in Section 3). We have been unable, however, to deduce one from the other.

3. Examples

In this section we will compute $l(P/fT)$ for certain height two prime ideals P of $T = K[[U, V, W]]$ where K is an infinite field. The motivation is two-fold. We want to identify more examples of prime ideals in hypersurface rings whose analytic spreads and heights are the same, and we also wish to produce examples to which Theorem 2.5 applies. All our examples are monomial primes of T. Recall that a prime ideal P of $K[X_1, ..., X_m]$ ($K[[X_1, ..., X_m]]$) is called a monomial prime if it is the kernel of the homomorphism $K[X_1, ..., X_m] \to K[s^{n_1}, ..., s^{n_m}]$ ($K[[X_1, ..., X_m]] \to K[[s^{n_1}, ..., s^{n_m}]]$) given by $X_i \to s^{n_i}$ where s is an indeterminate and $n_1 < n_2 < \cdots < n_m$. In [He] Herzog computed generators for all monomial primes when $m = 3$. We will use some facts about multiplicity in our proofs so we recall some notation. For a local ring (T, N), an ideal I which is primary to N, and a T-module L, we denote the multiplicity of L with respect to I by $e_T(I, L)$ (see [N, Chap. 23]). We also will use the symbol $\lambda_T(-)$ to denote the length of a T-module (e.g., $\lambda_T(T/I) =$ length of the T-module T/I).

The computational techniques used below are essentially the same as those used by C. Huneke in [H1]. We have adapted the ideas to attack our problem of computing $l(P/fT)$.

EXAMPLE 3.1. Let $T = K[[U, V, W]]$ and let P be the kernel of the homomorphism $T \to K[[s^{2k+1}, s^{2k+2}, s^{2k+3}]]$ where k is any positive integer. Then there exists $f \in P^{(2)}$ such that $l(P/fT) = 1$ and $r(P/fT) = 1$. Consequently, $\bigoplus P^{(n)}t^n$ is Noetherian.

Proof. We first claim that $P = (a, b, c)T$ where a, b, and c are the (sign alternating) maximal minors of the matrix

$$A = \begin{pmatrix} W & U^{k+1} & V \\ V & W^k & U \end{pmatrix}.$$

Although this can likely be deduced from Herzog's work (see [K, p. 137]), we will instead use a result about multiplicities from [N] (we are grateful to C. Huneke for suggesting this route). Let $I = (a, b, c)T$. Since I is defined by the minors of A it follows by [B] that the projective dimension of I is one. Therefore depth $(T/I) = 1$ by the Auslander–Buchsbaum formula; hence T/I is Cohen–Macaulay (note that $\dim(T/I) = 1$ since height$(I) = 2$). Furthermore, the image of U modulo I is a system of parameters for T/I. Thus $e_T(U, T/I) = \lambda_T(T/(U, I)T)$. Using that $a = U^{k+2} - VW^k$, $b = V^2 - UW$, and $c = W^{k+1} - U^{k+1}V$ we see that $T/(U, I)T = T/(U, VW^k, V^2, W^{k+1})T$. Hence, $\lambda_T(T/(U, I)T)$ = the number of monomials in U, V, W which are not contained in $(U, VW^k, V^2, W^{k+1})T$. A routine calculation shows that this number is $2k+1$; hence $e_T(U, T/I) = 2k+1$. By applying [N, Theorem 23.5] we have $e_T(U, T/I) = \sum e_T(U, T/Q) \cdot \lambda_{(T/I)_Q}(T_Q/IT_Q)$ where Q runs over all associated primes of

I such that $\dim(T/Q) = \text{height}(UT) = 1$ (i.e., all primes minimal over I). Since s^{2k+1} is a reduction of the maximal ideal of T/P, we have $e_T(U, T/P) = e_T(T/P) =$ multiplicity if T/P with respect to the maximal ideal. In addition, it is known that $e_T(T/P) = 2k+1$. Therefore $e_T(U, T/I) = e_T(U, T/P)$ which implies by the formula that P is the *only* prime minimal over I, and $IT_P = PT_P$. It now follows (since $\text{Ass}(T/I) = \{P\}$) that $I = P$.

Next we construct the element f. By the Hilbert–Burch Theorem [B] a minimal resolution for P is given by

$$0 \longrightarrow T^2 \xrightarrow{A} T^3 \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P \longrightarrow 0$$

and hence the relations on a, b, and c are generated by

(1) $aW + bU^{k+1} + cV = 0$,

(2) $aV + bW^k + cU = 0$.

Multiplying (1) by bW^{k-1} and (2) by a and eliminating abW^k, we obtain $U(b^2U^kW^{k-1} - ac) = V(a^2 - bcW^{k-1})$. Since $\{U, V\}$ is a T-regular sequence there exists $f \in T$ such that

(3) $Uf = a^2 - bcW^{k-1}$,

(4) $Vf = b^2U^kW^{k-1} - ac$.

If we instead multiply (1) by c and (2) by bU^k and eliminate bcU^{k+1}, we obtain

(5) $Wf = c^2 - abU^k$.

By either (3), (4), or (5) it follows that $f \in P^{(2)}$. Furthermore, if we let b_1 and P_1 denote the images of b and P modulo fT, then $b_1P_1 = P_1^2$ as a straight calculation using (3), (4), and (5) shows. Therefore $l(P/fT) = 1$ and $r(P/fT) \leqslant 1$. Since P/fT is not principal, $r(P/fT) = 1$. Finally, by Theorem 2.5 $\bigoplus P^{(n)}t^n$ is Noetherian. ∎

EXAMPLE 3.2. Let $T = K[[U, V, W]]$ and let P be the kernel of the homomorphism $T \to K[[s^{3k+1}, s^{3k+2}, s^{3k+4}]]$ where k is any positive integer. Then there exists $g \in P^{(3)}$ such that $l(P/gT) = 1$ and $r(P/gT) \leqslant 2$. Consequently $\bigoplus P^{(n)}t^n$ is Noetherian.

Proof. As in Example 3.1 we first find a generating set for P. We claim that $P=(a, b, c)T$ where a, b, and c are the maximal minors of the matrix

$$A=\begin{pmatrix} W & U^k & V \\ V^2 & W^k & U^2 \end{pmatrix}.$$

The proof of this claim is similar to the corresponding proof given in Example 3.1; hence we omit the details. By the Hilbert–Burch Theorem [B] we know that

$$0 \longrightarrow T^2 \xrightarrow{A} T^3 \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P \longrightarrow 0$$

is a minimal resolution of P; hence

(1) $aW+bU^k+cV=0$,

(2) $aV^2+bW^k+cU^2=0$ generate the relations on a, b, and c. Assume $k>1$ (we will do the case $k=1$ separately). Multiplying (1) by bW^{k-1} and (2) by a, and then eliminating abW^k, we obtain

$$U^2(b^2U^{k-2}W^{k-1}-ac)=V(a^2V-bcW^{k-1}).$$

Since $\{U^2, V\}$ is a T-regular sequence there exists $f \in T$ such that

(3) $U^2f=a^2V-bcW^{k-1}$,

(4) $Vf=b^2U^{k-1}W^{k-1}-ac$.

Furthermore, by considering VWf and using the relations $bW^k=-aV^2-cU^2$ and $aW=-bU^k-cV$, we find

(5) $Wf=c^2-abU^{k-2}V$.

Although $f \in P^{(2)}$, we cannot conclude from (3), (4), and (5) alone that $l(P/fT)=1$. We can, however, continue this computational process. Multiplying (3) by c and using (1) yields the equation

$$U^2(cf+a^2bU^{k-2})=W(-a^3-bc^2W^{k-2})$$

(we are using here that $k \geqslant 2$). Thus there exists $g \in T$ such that

(6) $U^2g=-a^3-bc^2W^{k-2}$

(7) $Wg=cf+a^2bU^{k-2}$.

In addition, if we multiply (3) by a and use (1) we obtain another equation involving g;

(8) $Vg = b^2cU^{k-2}W^{k-2} - af.$

In order to find more relations modulo g on the cubic terms of P we work with (7) and (8). Multiplying (7) by W and using (5), multiplying (7) by V and using (4), and multiplying (8) by V and using (4), it follows that

(9) $W^2g = -abcU^{k-2}V + c^3 + a^2bU^{k-2}W,$

(10) $VWg = b^2cU^{k-2}W^{k-1} - ac^2 + a^2bU^{k-2}V,$

(11) $V^2g = b^2cU^{k-2}W^{k-2}V - ab^2U^{k-2}W^{k-1} + a^2c.$

Notice that by (6), (9), (10) or (11), $g \in P^{(3)}$. Furthermore $(a, c)^3 \subset bP^2$ modulo (gT). If we denote by b_1 and P_1 the images of b and P modulo gT, it then follows that $b_1P_1^2 = P_1^3$. Therefore $l(P/gT) = 1$ and $r(P/gT) \leqslant 2$. For the case $k = 1$ one can perform a similar computation. The matrix A is given by

$$A = \begin{pmatrix} W & U & V \\ V^2 & W & U^2 \end{pmatrix}$$

so the relations are

$$aW + bU + cV = 0,$$
$$aV^2 + bW + cU^2 = 0.$$

The corresponding f satisfies

$$Uf = a^2V - bc,$$
$$Vf = b^2 - acU,$$
$$Wf = -abV + c^2U,$$

and the corresponding g satisfies

$$Wg = ab^2 + c^3,$$
$$U^2g = a^2cV - bc^2 - a^2bU,$$
$$UVg = b^2c - a^2bV - ac^2U,$$
$$V^2g = -b^3 + abcU - ac^2V.$$

If we let a_1 and P_1 denote the images of a and P modulo gT, it follows that $a_1P_1^2 = P_1^3$; hence $r(P/gT) \leqslant 2$ and $l(P/gT) = 1$. Furthermore, $g \in P^{(3)}$. Finally, by Theorem 2.5 $\bigoplus P^{(n)}t^n$ is Noetherian. ∎

481/128/2-4

EXAMPLE 3.3. Let $T=K[[U, V, W]]$ and let k be a positive integer. Let P be the kernel of the homomorphism $T \to K[[s^{3k+2}, s^{3k+3}, s^{3k+5}]]$. If $k \geqslant 3$ then there exists $g \in P^{(3)}$ such that $l(P/gT)=1$ and $r(P/gT) \leqslant 2$. If $k=1$ then there exists $f \in P$ such that $l(P/fT)=1$ and $r(P/fT) \leqslant 1$. Consequently, for $k \neq 2$ $\bigoplus P^{(n)}t^n$ is Noetherian.

Proof. By using an argument similar to that in Example 3.1 one can show that $P=(a, b, c)T$ where a, b, and c are the maximal minors of the matrix

$$A=\begin{pmatrix} W & U^{k+1} & V^2 \\ V & W^k & U^2 \end{pmatrix},$$

and that the relations on a, b, and c are given by

(1) $aW+bU^{k+1}+cV^2=0$,

(2) $aV+bW^k+cU^2=0$.

Assume $k \geqslant 3$. By eliminating terms in a way similar to Examples 3.1 and 3.2, it follows that there exists $f \in T$ such that

(3) $U^2f=-a^2+bcVW^{k-1}$,

(4) $Vf=ac-b^2U^{k-1}W^{k-1}$,

(5) $Wf=abU^{k-1}-c^2V$.

By doing further computations as in Example 3.2, there exists $g \in T$ such that

(6) $U^2g=af-bc^2W^{k-1}$,

(7) $Vg=-cf-ab^2U^{k-3}W^{k-1}$,

(8) $Wg=c^3+a^2bU^{k-3}$

(we are using here that $k \geqslant 3$). In order to find more cubic relations we multiply (6) by U^2 and use (3), multiply (7) by U^2 and use (3), and multiply (7) by V and use (4). The computations yield the relations

(9) $U^4g=-a^3+abcVW^{k-1}-bc^2U^2W^{k-1}$,

(10) $U^2Vg=a^2c-bc^2W^{k-1}V-ab^2U^{k-1}W^{k-1}$,

(11) $V^2g=-ac^2+b^2cU^{k-1}W^{k-1}-ab^2U^{k-1}VW^{k-1}$.

Any one of (8), (9), (10), or (11) implies that $g \in P^{(3)}$. Furthermore, the same equations show that $(a, c)^3 \subset bP^2$ modulo (gT). Thus, by letting b_1 and P_1 denote the images of b and P modulo gT it follows that $b_1P_1^2=P_1^3$. Therefore $l(P/gT)=1$ and $r(P/gT) \leqslant 2$. The case $k=1$ is similar and we omit the details. Theorem 2.5 implies that $\bigoplus P^{(n)}t^n$ is Noetherian for $k \neq 2$. ∎

We make several remarks concerning Examples 3.1, 3.2, and 3.3.

Remarks 3.4. (1) The examples examined by C. Huneke in [H1, Propositions 2.2, 2.3, and 2.4, and Corollary 2.2] also satisfy the hypotheses of Theorem 2.5. For [H1, Proposition 2.2], and hence [H1, Corollary 2.2], one can use the element d which is constructed in Huneke's proof to achieve the conditions set forth in the hypotheses of Theorem 2.5. For [H1, Propositions 2.3 and 2.4] more work is required. Consider the element (under Huneke's notation) d_{r+1}, constructed in the proof of [H1, Proposition 2.3]. By continuing the computations provided at the end of the proof of [H1, Proposition 2.3] one can show that

$$x^{r-1}y\, d_{r+1} \equiv a^{r-1}c^2 \text{ modulo } (bP^r)$$
$$x^{r-2}y^2\, d_{r+1} \equiv a^{r-2}c^3 \text{ modulo } (bP^r)$$
$$xy^{r-1}\, d_{r+1} \equiv ac^r \text{ modulo } (bP^r)$$
$$y^r d_{r+1} \equiv c^{r+1} \text{ modulo } (bP^r).$$

Combining these equations with $z\, d_{r+1} \equiv a^{r+1}$ modulo (bP^r), and $x^r\, d_{r+1} \equiv a^r c$ modulo (bP^r) (from Huneke's proof), it follows that $(a, c)^{r+1} \subset bP^r$ modulo (d_{r+1}), and $d^{r+1} \in P^{(r+1)}$. These facts imply the hypotheses of Theorem 2.5. For the example considered in [H1, Proposition 2.4], it follows similarly that $l(P/d_r T) = 1$, $r(P/d_r T) \leqslant r-1$, and $d_r \in P^{(r)}$, where d_r is the element constructed in the proof of that proposition.

(2) By considering $R = T/fT$ (or T/gT) in Examples 3.1, 3.2, and 3.3 we are also finding new examples of prime ideals in the hypersurface ring R satisfying the condition that the analytic spread equals the height. We observe that once an example of a prime ideal q of R satisfying $l(q) = \text{height}\,(q)$ is found, it follows quickly that such an example in a hypersurface *domain* also exists. The reasoning here is that if $l(P/fT) = \text{height}\,(P/fT)$ then $l(P/hT) = \text{height}\,(P/hT)$ for *any* factor h of f which is contained in P. Since T is a unique factorization domain we may factor f into a product of irreducible elements, and at least one of these must lie in P. By using this element we obtain a hypersurface domain (after going modulo the element). If R is a two-dimensionel hypersurface domain, $l(q) = \text{height}\,(q) = 1$, and q is not principal, then q is a singular prime of R (i.e., R_q is not regular). Thus, with Examples 3.1, 3.2, and 3.3 we are producing singular primes of a local hypersurface domain (after modifying if necessary to make R into a domain).

(3) By Proposition 2.2 it follows that f in Example 3.1 and g in Examples 3.2 and 3.3 are not superficial for P.

(4) The prime ideal $q = P/fT$ in Example 3.1 has the property that its associated garded ring is Cohen–Macaulay by [TI, Proposition 7.4]. Therefore $q^n = q^{(n)}$ for each $n \geqslant 1$ by [Hu2, Theorem 2.1].

References

[B] L. Burch, On ideals of finite homological dimension in local rings, *Proc. Cambridge Philos. Soc.* **64** (1968), 949–952.

[C] C. Cowsik, Symbolic powers and number of defining equations, *in* "Algebra and Its Applications," Lecture Notes in Pure and Applied Mathematics Series, Vol. 91, Dekker, New York, 1984.

[CN] R. C. Cowsik and Nori, On the fibers of blowing up, *J. Indian Math. Soc. (N.S.)* **40** (1976), 217–222.

[E] S. Eliahou, Symbolic powers of monomial curves, *J. Algebra* **117** (1988), 437–456.

[He] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, *Manuscripta Math.* **3** (1970), 175–193.

[Hu1] S. Huckaba, On linear equivalence of the P-adic and P-symbolic topologies, *J. Pure Appl. Algebra* **46** (1987), 179–185.

[Hu2] S. Huckaba, Symbolic powers of prime ideals with applications to hypersurface rings, *Nagoya Math. J.* **113** (1989), 161–172.

[H1] C. Huneke, On the finite generation of symbolic blow-ups, *Math Z* **179** (1982), 465–472.

[H2] C. Huneke, Hilbert functions and symbolic powers, *Michigan Math. J.* **34** (1987), 293–318.

[H3] C. Huneke, On the associated graded ring of an ideal, *Illinois J. Math.* **26** (1982), 121–137.

[KR] D. Katz and L. J. Ratliff, Jr., On the symbolic Rees ring of a primary ideal, *Comm. Algebra* **14** (1986), 959–970.

[K] E. Kunz, "Introduction to Commutative Algebra and Algebraic Geometry," Birkhäuser, Boston, 1985.

[L] J. Lipman, Relative Lipschitz-saturation, *Amer. J. Math.* **97** (1975), 791–813.

[Mat] H. Matsumura, "Commutative Algebra," 2nd ed., Benjamin/Cummings, Reading, MA, 1980.

[M] S. McAdam, "Asymptotic Prime Divisors," Lecture Notes in Math., Vol. 1023, Springer-Verlag, New York/Berlin, 1983.

[N] M. Nagata, "Local rings," Krieger, Huntington/New York, 1975.

[NR] D. G. Northcott and D. Rees, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.

[O] A. Ooshi, Noetherian property of symbolic Rees algebras, *Hiroshima Math. J.* **15** (1985), 581–584.

[R] L. J. Ratliff, Jr., The topology determined by the symbolic powers of primary ideals, *Comm. Algebra* **13** (1985), 2073–2104.

[Ro] P. Roberts, A prime ideal in a polynomial ring whose ssymbolic blow-up is not Noetherian, *Proc. Amer. Math. Soc.* **94** (1985), 589–592.

[S] P. Schenzel, Symbolic powers of prime ideals and their topology, *Proc. Amer. Math. Soc.* **93** (1985), 15–20.

[TI] N. V. Trung and S. Ikeda, When is the Rees algebra Cohen–Macauley?, preprint.

[ZS] O. Zariski and P. Samuel, "Commutative Algebra," Vol. II, Springer-Verlag, New York, 1960.